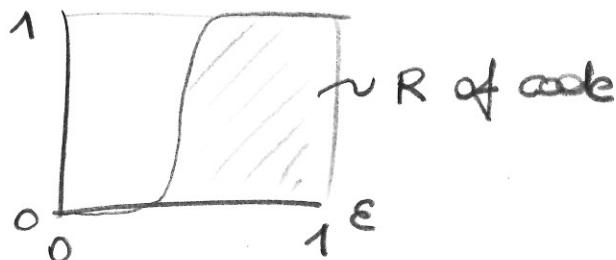


①

# Capacity Via Symmetry II

## Generalization:

- Recall: BEC \*  $h_i(\epsilon) = H(X_i | Y_{\setminus i}(\epsilon))$  \* if we can show that it has a sharp threshold, we are done
- \* fulfills Area theorem
  - \* has sharp threshold



What quantity should I consider for the general case?  
There are many options; there are a few meaningful ones,

$\epsilon$  here is entropy ( $\epsilon = h_2(p)$  for BSC)

- \*  $h_i(\epsilon) = H(X_i | Y_{\setminus i}(\epsilon))$  EXIT function; in general it does not fulfill area theorem
- \*  $g_i(\epsilon) = \frac{\partial H(X_i | Y_{\setminus i}(\epsilon))}{\partial \epsilon_i} \Big|_{\epsilon_i=\epsilon}$  EXIT function; does fulfill area theorem
- \*  $P_{i,b} = \Pr[\hat{X}_i^{\text{MAP}}(Y_{\setminus i}) = X_i]$

Does it make a difference which quantity we consider?

(2)

It turns out that they are all equivalent for our purpose.

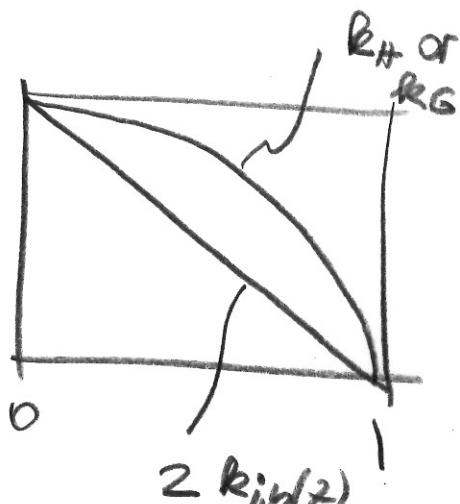
To see this, it is best to look at the posterior density  $p(x_i | y_i)$  in the so called IDI domain. All the quantities we mentioned are linear functions of this density, with the following kernels. Note that all those kernels are concave and decreasing.

$$k_{i,b}(z) = \frac{1}{2}(1-z)$$

$$k_{\#}(z) = h\left(\frac{1-z}{2}\right)$$

$$k_G(z, p) = \begin{array}{l} \text{more complicated} \\ \text{but explicit} \end{array}$$

↑  
transition probability



From this we get that we can bound quantities in all possible ways. E.g.)

$k_G^{-1}(g_i)$	$\leq 2P_{b,i} \leq g_i$
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(3)

From this it follows that as long as we can show that any of these quantities has a sharp threshold at a certain channel parameter, the EXIT also jumps there and we are done.

What channel should we consider?  
It turns out that if we could prove a sufficiently strong result for the BSC then we are in fact done.

Theorem (Sasoglu 2011)

The BSC is the least capable channel of all BMS channels of a fixed capacity.

Theorem (Sasoglu 2011) Let A and B be two BMS channels of capacity C. Let A be more capable and consider block-MAP decoding. Then

$$P_A \leq N P_B + h_2(P_b)$$

(4)

So this is a priori good. For the BSC we have a very similar "combinatorial" set-up, where  $\mathcal{L}_i$  is lets say the set of error patterns (excluding position  $i$ ) that cause trouble.

So why are we not done?

Problem:  $\mathcal{L}_i$  is in general not monotone!

Let  $z_{n_i}$  be the error pattern and assume without loss of generality transmission of the all-zero codeword

$$L_i(z_{n_i}) = \log \frac{\sum_{c \in \mathcal{E}_{i,0}} p(z_{n_i}|c)}{\sum_{c \in \mathcal{E}_{i,1}} p(z_{n_i}|c)}$$

Assume I put  $z_{n_i} \in \mathcal{L}_i$  iff  $L_i(z_{n_i}) \leq 0$ .  
But this is unfortunately not monotone!

(5)

So let us look at a simpler problem for the moment. What is the relationship between

$$P_b = \frac{1}{n} \sum P_{i,b} \quad \text{extrinsic bit-MAP}$$

$$\tilde{P}_b = \frac{1}{n} \sum \tilde{P}_{i,b} \quad \text{bit-MAP}$$

$$P_B$$

$$\text{block MAP}$$

For the BEC we know that  $P_b \xrightarrow{n \rightarrow \infty} 0$  arbitrarily close to the Shannon threshold.

First note that  $\tilde{P}_b = \epsilon P_b$  (only a constant different)

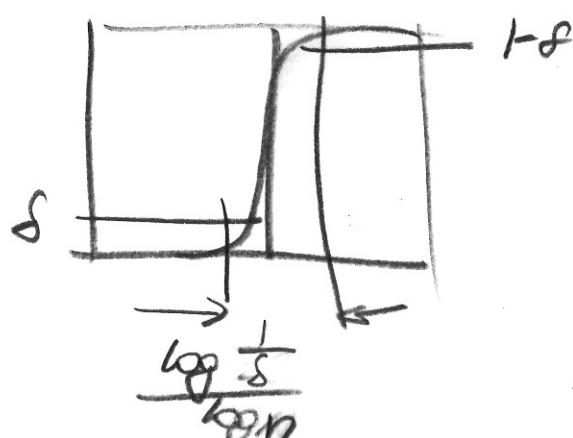
$\uparrow$  channel parameter  
(for the general case very similar);

extremes of info combining!

So also the true bit-MAP error probability converges to 0 arbitrarily close to the Shannon threshold. But in general

$$\tilde{P}_b \leq P_B \leq n \tilde{P}_b$$

Unfortunately



(for the simple  
Kolai, Friedewal  
result)

(6)

So we cannot let  $S = \frac{1}{n^\alpha}$  for  $\alpha > 1$   
as necessary.

Two possible solutions.

(I) [Kumar, Pfister] Use a stronger sharp threshold result by Bourgoin et al.

$$gap \sim \frac{\log \frac{1}{S}}{w_n(\log n)}, \text{ where } w_n \xrightarrow{n \rightarrow \infty} \infty$$

Now we can pick e.g.  $S = \frac{1}{n^2}$ ,

(II) Use weight distributions and the union bound.

- + works for any channel
- perhaps not quite as elegant

(7)

Think of BEC. Let  $C$  be a code with weight distribution  $A(x) = \sum A_w x^w$ . Let  $\epsilon$  be the channel parameter. Then

$$P_B \leq A(\epsilon) \quad \text{This is the union bound.}$$

This bound is in general not tight. But if for a fixed  $\epsilon$

$$\sum_{w=1}^W A_w \epsilon^w$$

is small then we know that it is unlikely that a decoder makes a mistake to a codeword of weight  $w$  or smaller.

This is what we will use. We will show that the block MAP decoder is very unlikely to make a mistake to a word of weight less than  $N^{(1-\epsilon)}$ . This follows by bounding the weight distribution and by using the above union bound. But it also cannot make a mistake to words larger than this since this would give rise to a bit error probability of at least  $N^{-\frac{1}{2}}$ . But this contradicts our previous result.

(8)

Let us look a little bit more in detail at the weight distribution.

We need an upper bound as the weight distribution for RM codes.

$$R = 2^{-n} \sum_{i=0}^v \binom{n}{i} \quad \text{so } v \approx \frac{n}{2} \pm \sqrt{n}$$

This is a classical problem.

SB70 Sloan Berlekamp

KTA76 Kasami Tokura Azumi

KLPI2 Keulen, Lovett, Porat

ASW14 Abbe Shpilka Widgerson

$$A(n, v) \leq 2^{c(v+2)^2(ne + \sum_{i=0}^{v-e} \binom{n-e}{i})}$$

$\nearrow$   
log<sub>2</sub> N order

$$w = 2^{n-e-1}$$